Propagation of Waves in a Plasma in a Magnetic Field*

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Summary—**The propagation of electromagnetic waves in a plasma in a magnetic field as given by the Appleton-Hartree theory is discussed in terms of the wave normal surfaces instead of the more conventional propagation vector plots, and the "ordinary" and "extraordinary" waves are defined in terms of their polarizations instead of using a continuity argument. This gives a different picture of "a wave" which has some advantages. In particular, "whistlers" become obvious, as are regions of high reflection and high absorption.**

The Appleton-Hartree theory is then extended to include the effect of electron temperature, and this results in a third wave whose velocity is of the order of electron thermal motions.

THE propagation of electromagnetic waves through an ionized gas in a magnetic field has been studied[1] since Appleton and Hartree derived the magneto-ionic equations, and most of what is said here can be obtained from their equations although taken from a somewhat different point of view. The remainder comes from the Russian literature,[2] although that has to be reworked considerably to be understood. The medium to be discussed is a collisionless plasma in which the ions are assumed to be stationary.

Let us first discuss propagation in a plasma without a magnetic field (Fig. 1). It differs from that in a dielectric in that bound electrons generally oscillate in phase with the applied force, whereas a free electron oscillates out of phase. Consequently, the phase velocity u of an electromagnetic wave in a dielectric is slower than the velocity c in free space, whereas it is faster in a plasma. This does not violate relativity as the group velocity w is always less than c.

It is conventional to represent the dispersion of electromagnetic waves by a plot of the propagation constant k against ω, and for a field free plasma this gives a hyperbola which "cuts off" below the plasma frequency $\omega_p^2 = Ne^2/\epsilon_0 m$ (Fig. 2). Consequently, radio waves of frequency less than the ω_p appropriate for the ionosphere are reflected back to earth. In terms of the ratio

$$\alpha^2 = \frac{\omega_p^2}{\omega^2} = \frac{\text{Kinetic Energy}}{\text{Field Energy}} = \frac{Nmv^2}{\epsilon_D E^2}$$

the index of refraction $n = c/u$ is given by

$$n^2 = \frac{c^2}{u^2} = \frac{k^2c^2}{\omega^2} = 1 - \alpha^2.$$

The symbols α^2 and β which are used correspond to X and Y in the accepted URSI international notation.

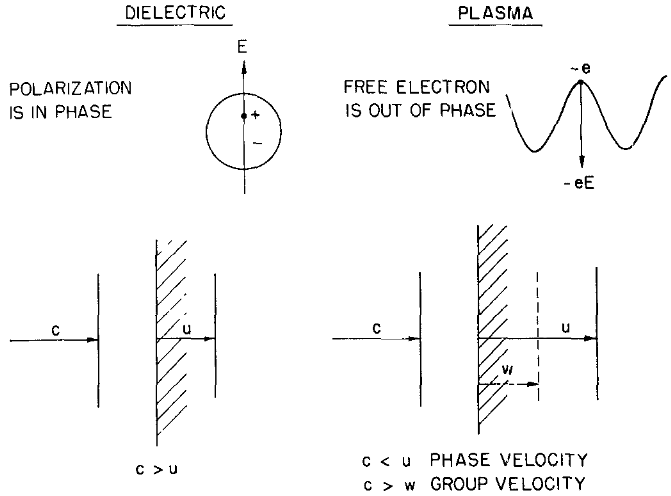

Fig. 1—Contrast of polarization due to bound and free electrons.

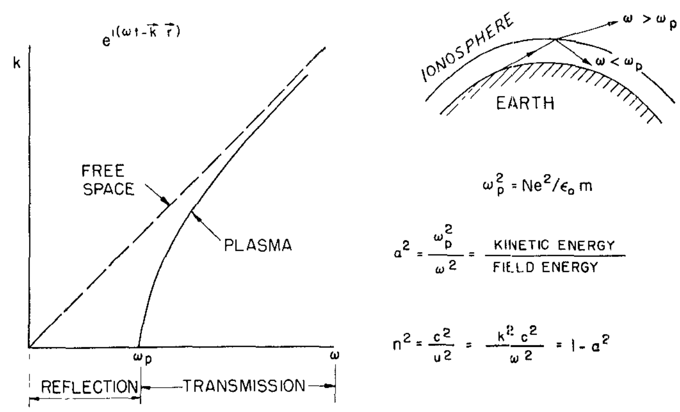

Fig. 2—Propagation constant plot for a plasma without magnetic field.

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[1] J. A. Ratcliffe, "The Magneto-Ionic Theory and Its Applications to the Ionosphere," Cambridge University Press, London, Eng.; 1959. This work provides most of the pertinent references.

[2] A. G. Sitenko and K. N. Stepanov, *Soviet Phys., JETP*, vol. 4, pp. 512–520; 1957.

In the presence of a magnetic field, the conductivity σ becomes a tensor $\boldsymbol{\sigma}$; and from that, one obtains an equivalent dielectric tensor $\boldsymbol{K}$,

$$\boldsymbol{K} = 1 + \frac{\boldsymbol{\sigma}}{j\omega\epsilon_0} = \begin{vmatrix} K_T & K_H & 0 \\ -K_H & K_T & 0 \\ 0 & 0 & -K_P \end{vmatrix} \qquad \begin{aligned} K_r &= 1 - \frac{\alpha^2}{1-\beta} \\ K_l &= 1 - \frac{\alpha^2}{1+\beta} \\ 2K_T &= K_r + K_l \\ 2jK_H &= K_r - K_l. \end{aligned}$$

which turns out to have only three distinct components, K_{Parallel}, $K_{\text{Transverse}}$, and K_{Hall}—the subscripts referring to the directions of the electric field relative to the magnetic field. The components K_T and jK_H are the sum and difference of K_{right} and K_{left} which refer to circular polarizations of the electric field.

For a plane wave, the wave equation may be written

$$\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{E}) + \boldsymbol{K} \cdot \boldsymbol{E} = 0,$$

where $\boldsymbol{n}$ is a vector normal to the wave whose magnitude is the index of refraction $n = c/u$. This equation is linear homogeneous and hence the determinant of its coefficients must vanish. This yields the dispersion relation

$$\tan^2\theta = -\frac{K_p(n^2 - K_r)(n^2 - K_l)}{(n^2 - K_p)(K_T n^2 - K_r K_l)}.$$

It is convenient to have this equation solved for $\tan^2\theta$, as setting the numerator equal to zero gives the indices of the two waves (right- and left-handed) which can propagate along $\boldsymbol{B}_0$ and setting the denominator equal to zero gives the two waves (ordinary and extraordinary) which propagate across $\boldsymbol{B}$.

$$\text{Propagation Along } \boldsymbol{B}_0: \begin{cases} n_r^2 = K_r \\ n_l^2 = K_l \end{cases} \qquad \theta = 0$$

$$\text{Propagation Across } \boldsymbol{B}_0: \begin{cases} n_0^2 = K_p \\ K_T n_x^2 = K_r K_l \end{cases}. \qquad \theta = \pi/2$$

The polarizations of these four waves in the principal directions are shown in Fig. 3, as are the dispersion equations, which are expressed by giving the index of refraction as a function of $\alpha = \omega_p/\omega$ and $\beta = \omega_b/\omega$, $\omega_b = eB/m$. These relations can be shown by propagation constant plots in Fig. 4.

The *cutoff* for any of these waves occurs when $k = 0$ or $n = \infty$, and they have a *resonance* when $k = \infty$ or $n = 0$. These differ in that a wave in an inhomogeneous medium which is approaching a cutoff gets refracted away from the cutoff, whereas one approaching a resonance gets refracted towards it. The four singular situations are named in Table I.

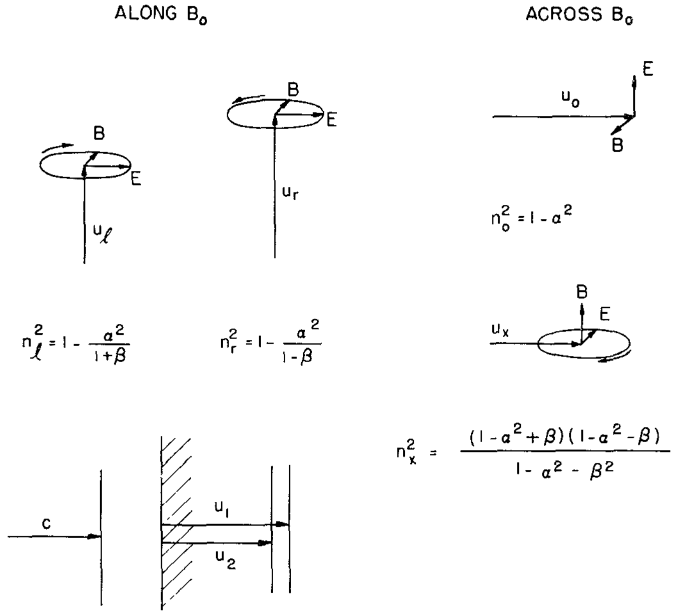

Fig. 3—Polarization of the principal waves.

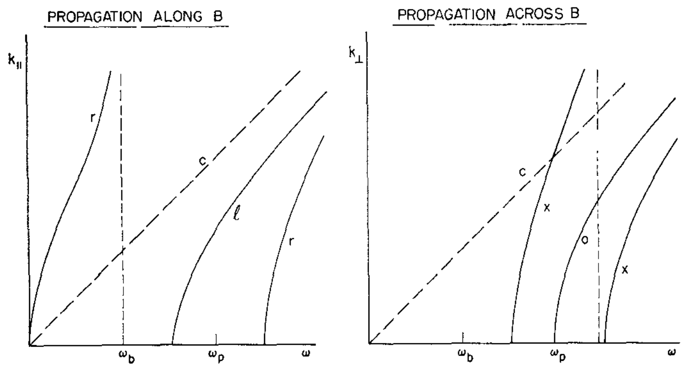

Fig. 4—Propagation constants in a magnetic field.

TABLE I

	Cutoff $u = \infty$	Resonance $u = 0$
Cyclotron:	$(\omega \pm \omega_b)^2 = \omega_b^2 + \omega_p^2$	$\omega^2 = \omega_b^2$
Plasma:	$\omega^2 = \omega_p^2$	$\omega^2 = \omega_p^2 + \omega_b^2$

These four equations are represented by lines, three straight and one parabolic, on a plot of β^2 against α^2 (Fig. 5). As n^2 changes sign when one crosses a cutoff or resonance line, n changes from real to imaginary, and the corresponding wave ceases to propagate. Accordingly, the four lines bound 8 areas on the α^2-β^2 plot, in each of which the set of principal waves which propagate is different. It is interesting to show not only the principal waves ($\theta = 0$ and $\theta = \pi/2$) but also the propagation at an arbitrary angle θ to the magnetic field, and this is done by plotting the velocity u against θ in a polar plot. Such a diagram is shown in the figure for

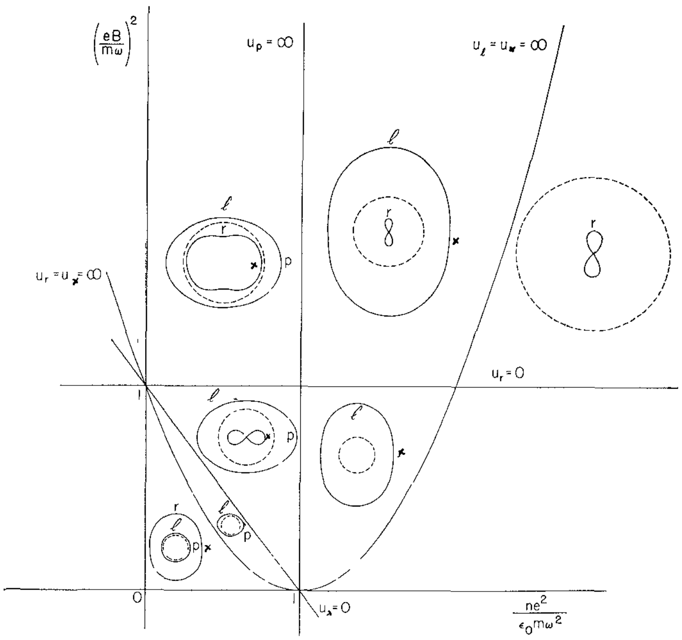

Fig. 5—Wave normal surfaces in a plasma in a magnetic field.

some representative point in each area. In each diagram, the magnetic field is directed up and the "wave normal surface" is obtained by rotating the diagram around this direction as an axis. Unfortunately, the scale of each diagram is different, and this is shown by the dotted circles which represent the velocity of light in free space for each diagram.

One sees that there is one area in the $\alpha^2-\beta^2$ plane at low magnetic fields ($\beta^2<1$) for which no wave can propagate two others at low fields where only one wave surface propagates; two areas at low densities ($\alpha^2<1$) where there is complete propagation in all directions; and three areas where the wave surface crosses itself at the origin, that is for which there is a resonant cone. When this cone has a small angle with respect to the magnetic field, waves are guided by the magnetic field and this is responsible for the phenomenon of "whistlers."

Consider the region of the $\alpha^2-\beta^2$ plot near the β^2 axis. If one performs experiments at increasing magnetic fields, which correspond to vertical displacements on the plot, with a right-hand polarized wave, one expects a sharp resonance absorption in the region where the r wave does not propagate. If one repeats the experiment at a higher plasma density (further to the right on the plot), one expects a resonance broadened on the high-frequency side because the area in which the r wave does not propagate is broader here. The calculated results with no collisions, and with few collisions, are shown in Fig. 6 (next page). Experimental results, taken both in emission and absorption, are shown in Fig. 7.

This completes what must be said here about the simple plasmas considered by Appleton and Hartree. We must now consider plasmas in which the thermal motions are not negligible, although we shall still consider them as small. The dielectric tensor now fills out so that there are six independent components

$$\boldsymbol{K} = 1 + \frac{\boldsymbol{\sigma}}{j\omega\epsilon_0} = \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ -K_{12} & K_{22} & K_{12} \\ K_{13} & -K_{23} & K_{33} \end{vmatrix} \qquad \alpha^2 = \frac{Ne^2}{m\epsilon_0\omega^2}, \quad \beta^2 = \left(\frac{eB}{m\omega}\right)^2$$

and each component is considerably more complicated. As an example,

$$K_{11} = 1 - \frac{\alpha^2}{1-\beta^2} - \epsilon n^2\left[\frac{1+3\beta^2}{(1-\beta^2)^3}\zeta^2 + \frac{3\xi^2}{(1-\beta^2)(1-4\beta^2)}\right]$$

where ξ and ζ are the direction cosines of k and the dispersion equation is now bi-cubic.

$$-\epsilon a n^6 + An^4 - Bn^2 + C = 0, \qquad \epsilon = \left(\frac{kT}{mc^2}\right)\alpha^2,$$

so that there are three wave surfaces. In general, two of these are substantially the same as the ones studied previously and will be called "electromagnetic." The third has a phase velocity of the order of electron speeds and will be called the "plasma electron" wave.

For propagation along $\boldsymbol{B}_0$ the three indices are

$$\text{Fast Waves}\begin{cases} n_r^2 = \dfrac{1-\alpha^2/(1-\beta)}{1+\epsilon/(1-\beta)^3} \\[2ex] n_l^2 = \dfrac{1-\alpha^2/(1+\beta)}{1+\epsilon/(1+\beta)^3} \end{cases}$$

$$\text{Plasma Wave } n_p^2 = \frac{1-\alpha^2}{3\epsilon}$$

The plasma wave n_p has the dispersion relation given by Vlasov and by Bohm and Gross and its propagation plot is given in Fig. 8.

For propagation across B_0 the dispersion is more complicated, the extraordinary electromagnetic and plasma waves being coupled in one equation as their polarizations are similar.

There is now a new resonance at $\beta^2=1/4$ or $\omega=2\omega_b$, so that there are now 13 regions on the $\alpha^2=\beta^2$ plot. The wave surfaces have been calculated by R. Papa and are shown in Figs. 9 and 10. The plasma wave surfaces are shown here considerably enlarged, as they are, in fact, generally much smaller than the electromagnetic waves.

When one also includes the ion temperature, a fourth wave will appear, but this has not been worked out in detail. There is an amazing variety to the possible waves in a plasma in a magnetic field.

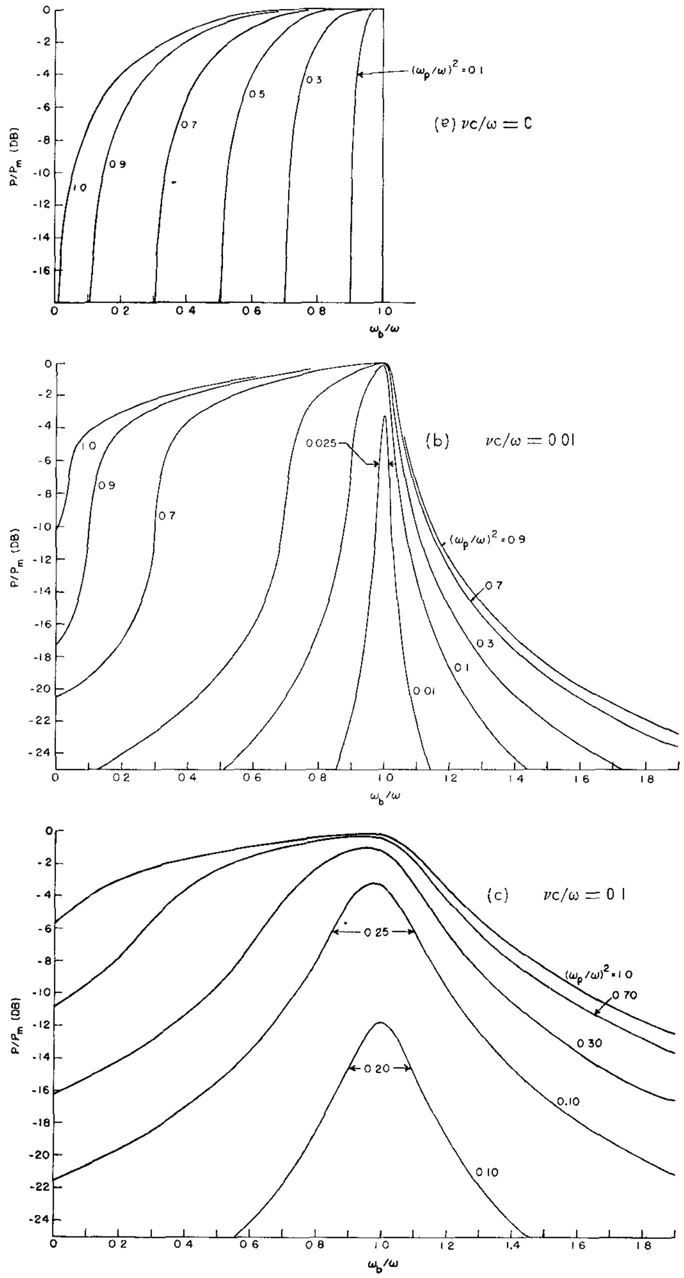

Fig. 6—Attenuation of a right-hand polarized wave transmitted through a plasma slab.

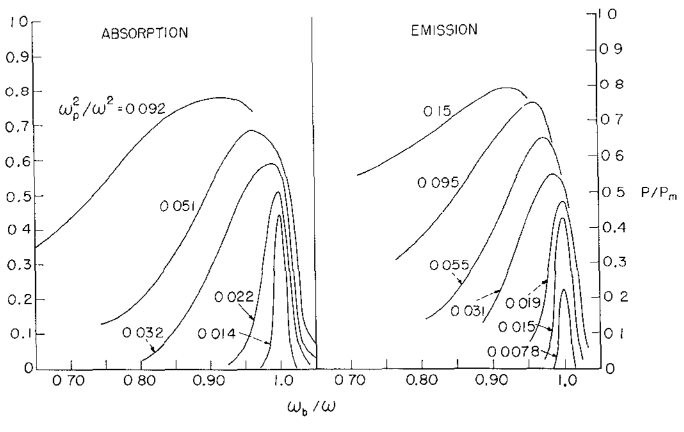

Fig. 7—Observed absorption and emission of microwaves from a plasma.

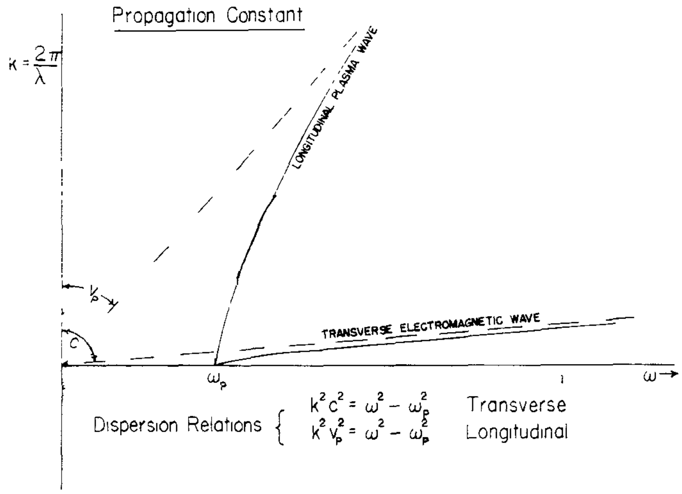

Fig. 8.

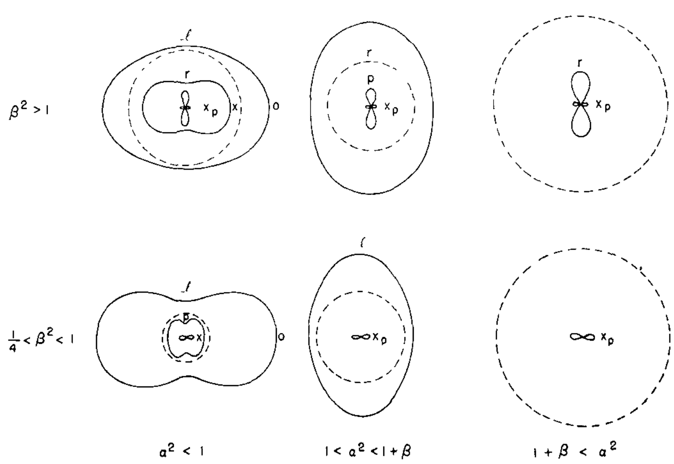

Fig. 9—Normal wave surfaces including electron temperatures.

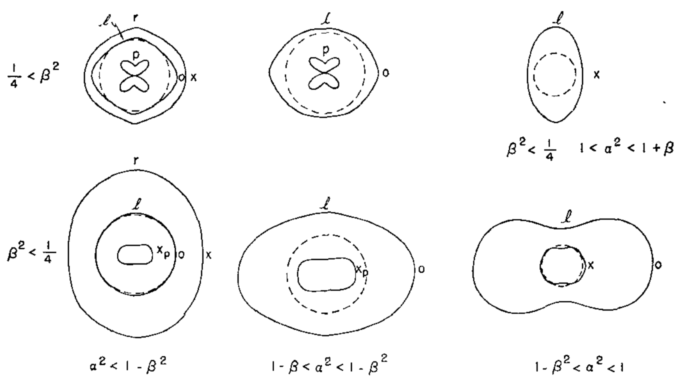

Fig. 10—Normal wave surfaces including electron temperatures.